Risk-Neutral Pricing
Part II - Eliminating The Arbitrage

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In Part I of this series we determined that given our matrix of asset prices at time zero and asset payoffs at time one there was an arbitrage available to us such that for an investment of $0 at time zero there was a certain payoff of $100 at time one regardless of the state-of-the-world at that time. In Part II of this series we will determine what asset prices at time zero prohibit arbitrage. To demonstrate the mathematics we will use the two state economy, state probabilities and discount rates from Part I, which are...

The table below presents time zero asset prices and time one asset payoffs depending on the state-of-the-world (state ω_a or state ω_b) at that time...

Table 1 - Our Two State Economy

Asset Symbol	Price t = 0	Payoff t = 1 ω_a	ω_b
B	100	105	105
X	80	120	60
Y	50	40	80

The table below presents time zero probabilities of finding ourselves in either state ω_a or state ω_b at time one...

Table 2 - State Probabilities (Measure P)

Description	Symbol	Probability
Probability that we will find ourselves in state ω_a at time one	p	0.60
Probability that we will find ourselves in state ω_b at time one	$1-p$	0.40

The table below present the risk-adjusted discount rates that in combination with Tables 1 and 2 above were used to determine the prices of assets B, X and Y at time zero...

Table 3 - Risk-Adjusted Discount Rates

Asset	Symbol	Discount Rate
B	K_b	0.05
X	K_x	0.20
Y	K_y	0.12

Our Hypothetical Problem

In Part I we determined that if we use the tables above we can construct a portfolio that consists of a short position of 8.99 units of Asset B, a long positions of 5.80 units of Asset X and a long position of 8.70 units of Assets Y such that the portfolio costs $0 to set up at time zero and results in a certain payoff of $100 at time one regardless of the state-of-the-world at that time. This is an arbitrage and is inconsistent with economic equilibrium.

Question: Given the time zero price of assets B and X, what does the time zero price of Asset Y have to be to prohibit arbitrage?

Finding The Arbitrage

In Part I we defined **matrix A** to be a matrix of asset prices at time zero and asset payoffs at time one where matrix row one represents the price and payoffs of Asset B, matrix row two represents the price and payoffs of Asset X and matrix row three represents the price and payoffs of Asset Y. Matrix A in matrix notation is...

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} (B_0) & (X_0) & (Y_0) \\ B_a & X_a & Y_a \\ B_b & X_b & Y_b \end{bmatrix} \tag{1}$$

In Part I we defined **vector v** to be a vector of units where θ_b, θ_x and θ_y represent the number of units of Asset B, Asset X and Asset Y, respectively, that are either purchased (long position) or sold (short position) at time zero. Note that this is a vector of unknowns that we solve for given asset prices at time zero, asset payoffs at time one and arbitrage portfolio payoffs at time one. Vector v in vector notation is...

$$\vec{\mathbf{v}} = \begin{bmatrix} \theta_b \\ \theta_x \\ \theta_y \end{bmatrix} \tag{2}$$

In Part I we defined **vector u** to be a vector of arbitrage portfolio cash flows where vector element p_1 is the cash flow associated with the cost of setting up the arbitrage portfolio at time zero, vector element p_2 is the arbitrage portfolio payoff at time one given state ω_a and vector element p_3 is the arbitrage portfolio payoff at time one given state ω_b. Note that an arbitrage exists if $p_1 = 0$, $p_2 \geq 0$, $p_3 \geq 0$ and $p_2 + p_3 > 0$. Vector u in vector notation is...

$$\vec{\mathbf{u}} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix} \tag{3}$$

In Part I we determined that the solution to vector v was the matrix:vector product of the inverse of matrix A and vector u. This solution in matrix:vector product notation is...

$$\mathbf{A}^{-1}\vec{\mathbf{u}} = \vec{\mathbf{v}} \tag{4}$$

Note that if matrix A **can be inverted** then there is a unique solution to Equation (4) above, which means that the arbitrage as defined by vector u exists. If matrix A **cannot be inverted** then...

(1) There is no solution to Equation (4). If there is no solution then there are no trading strategies that will result in the arbitrage as defined by vector u. This arbitrage does not exist.

(2) There are an infinite number of solutions to Equation (4). If there are an infinite number of solutions then there are an infinite number of trading strategies that will result in the arbitrage as defined by vector u. This arbitrage exists and can be exploited in an infinite number of ways.

The Plan - To eliminate the arbitrage we will construct matrix A such that it cannot be inverted. We will then determine if there is no solution or an infinite number of solutions to Equation (4). If there is no solution then the arbitrage as defined by vector u does not exist. We will also show that through this construction of matrix A any arbitrage (and not only the arbitrage as defined by vector u) is prohibited.

No-Arbitrage Mathematics

When pricing our assets at time zero we utilized the asset payoffs in Table 1, the state probabilities in Table 2 and the risk-adjusted discount rates in Table 3. We will define B_T, X_T and Y_T to be the payoffs on assets B, X and Y, respectively, at time one given the state-of-the-world at that time (which is known at time one but unknown at time zero). The equations for the prices of our three assets at time zero utilizing probability Measure P (as defined by Table 2) are...

$$B_0 = \mathbb{E}^P\left[B_T \times \left(1 + K_b\right)^{-1}\right] = \left(B_a\, p + B_b\,(1-p)\right)\left(1 + K_b\right)^{-1} \tag{5}$$

$$X_0 = \mathbb{E}^P\left[X_T \times \left(1 + K_x\right)^{-1}\right] = \left(X_a\, p + X_b\,(1-p)\right)\left(1 + K_x\right)^{-1} \tag{6}$$

$$Y_0 = \mathbb{E}^P\left[Y_T \times \left(1 + K_y\right)^{-1}\right] = \left(Y_a\, p + Y_b\,(1-p)\right)\left(1 + K_y\right)^{-1} \tag{7}$$

In Part I we determined that the equation for the determinant of matrix A was..

$$|A| = -B_0\left(X_a Y_b - X_b Y_a\right) + X_0\left(B_a Y_b - B_b Y_a\right) - Y_0\left(B_a X_b - B_b X_a\right) \tag{8}$$

If we substitute asset pricing Equations (5), (6) and (7) into Equation (8) the equation for the determinant of matrix A becomes...

$$\begin{aligned} |A| = &-\left(B_a\, p + B_b\,(1-p)\right)\left(X_a Y_b - X_b Y_a\right)\left(1 + K_b\right)^{-1} + \left(X_a\, p + X_b\,(1-p)\right)\left(B_a Y_b - B_b Y_a\right)\left(1 + K_x\right)^{-1} \\ &- \left(Y_a\, p + Y_b\,(1-p)\right)\left(B_a X_b - B_b X_a\right)\left(1 + K_y\right)^{-1} \end{aligned} \tag{9}$$

To eliminate the arbitrage we will construct matrix A such that it cannot be inverted. A matrix cannot be inverted when the matrix determinant is equal to zero. Through inspection of Equation (9) we see that most, if not all, of the right side of that equation can be eliminated if the discount rates were equal (i.e. $K_b = K_x = K_y$). If we force the discount rates to be equal then we must adjust the probability measure such that the asset prices obtained in Equations (5), (6) and (7) do not change.

The table below presents the time zero **risk-neutral** probabilities of finding ourselves in either state ω_a or state ω_b at time one...

Table 4 - State Probabilities (Measure Q)

Description	Symbol	Probability
Risk-neutral probability that we will find ourselves in state ω_a at time one	q	To be determined
Risk-neutral probability that we will find ourselves in state ω_b at time one	$1-q$	To be determined

Using the risk-neutral probablity Measure Q we can rewrite Equations (5), (6) and (7), respectively, as...

$$B_0 = \mathbb{E}^Q\left[B_T \times \left(1 + K_b\right)^{-1}\right] = \left(B_a\, q + B_b\,(1-q)\right)\left(1 + K_b\right)^{-1} \tag{10}$$

$$X_0 = \mathbb{E}^Q\left[X_T \times \left(1 + K_b\right)^{-1}\right] = \left(X_a\, q + X_b\,(1-q)\right)\left(1 + K_b\right)^{-1} \tag{11}$$

$$Y_0 = \mathbb{E}^Q\left[Y_T \times \left(1 + K_b\right)^{-1}\right] = \left(Y_a\, q + Y_b\,(1-q)\right)\left(1 + K_b\right)^{-1} \tag{12}$$

Rather than define matrix A asset prices at time zero in terms of the actual probability Measure P we will define matrix A asset prices at time zero in terms of the risk-neutral probability Measure Q. If we substitute Appendix Equations (33), (34) and (35) into Equation (8) the equation for the determinant of matrix A becomes...

$$\begin{aligned} |A| = &-\left(B_a X_a Y_b\, q - B_a X_b Y_a\, q + B_b X_a Y_b - B_b X_b Y_a - B_b X_a Y_b\, q + B_b X_b Y_a\, q\right)\left(1 + K_b\right)^{-1} + \left(B_a X_a Y_b\, q - B_b X_a Y_a\, q\right. \\ &\left. + B_a X_b Y_b - B_b X_b Y_a - B_a X_b Y_b\, q + B_b X_b Y_a\, q\right)\left(1 + K_b\right)^{-1} - \left(B_a X_b Y_a\, q - B_b X_a Y_a\, q + B_a X_b Y_b - B_b X_a Y_b\right. \\ &\left. - B_a X_b Y_b\, q + B_b X_a Y_b\, q\right)\left(1 + K_b\right)^{-1} \end{aligned} \tag{13}$$

Note that all of the terms on the right hand side of Equation (13) cancel such that the equation for the determinant of matrix A under the risk-neutral probablity Measure Q becomes...

$$|A| = 0 \tag{14}$$

The first part of our plan was to construct matrix A such that it cannot be inverted. When we construct matrix A using the risk-neutral probability Measure Q the result is a matrix determinant equal to zero, which means that the matrix cannot be inverted. The first part of our plan has been accomplished!

No Solution Or An Infinite Number Of Solutions

If we price the assets in our economy using the risk-neutral probability Measure Q and a discount rate equal to the risk-free rate (K_b) then the determinant of matrix A is zero, which means that matrix A cannot be inverted. If matrix A cannot be inverted then there are either (1) no solutions to our arbitrage or (2) an infinite number of solutions to our arbitrage. We will now determine if we have no solutions (an arbitrage does not exist) or an infinite number of solutions (an arbitrage exists and can be exploited an infinite number of ways).

We will use matrix A (Equation (1)) and vector u (Equation (3)) to construct the augmented matrix A'. The equation for the augmented matrix is...

$$\mathbf{A}' = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & p_1 \\ a_{21} & a_{22} & a_{23} & p_2 \\ a_{31} & a_{32} & a_{33} & p_3 \end{array}\right] \tag{15}$$

We will now perform row operations to put the augmented matrix (as defined by Equation (15) above) into row echelon form. To this end we will make the following definitions...

$$\alpha_1 = \frac{a_{21}}{a_{11}} = \frac{a_{31}}{a_{11}} = -\frac{B_T}{B_0} \text{ ...and... } \alpha_2 = \frac{a_{32} - \alpha_1\, a_{12}}{a_{22} - \alpha_1\, a_{12}} = \frac{X_b - \alpha_1(-X_0)}{X_a - \alpha_1(-X_0)} = \frac{X_b + \alpha_1 X_0}{X_a + \alpha_1 X_0} \tag{16}$$

To make augmented matrix element [R2C1] (R is row and C is column) equal to zero we perform the following row operation...

$$R_2 = R_2 - \frac{a_{21}}{a_{11}}\, R_1 = R_2 - \alpha_1\, R_1 \tag{17}$$

After performing the row operation in Equation (17) above the augmented matrix row 2 becomes...

$$[R2C1] = 0 \text{ ...and... } [R2C2] = a_{22} - \alpha_1\, a_{12} \text{ ...and... } [R2C3] = a_{23} - \alpha_1\, a_{13} \text{ ...and... } [R2C4] = p_2 - \alpha_1\, p_1 \tag{18}$$

To make augmented matrix element [R3C1] equal to zero we perform the following row operation...

$$R_3 = R_3 - \frac{a_{31}}{a_{11}}\, R_1 = R_3 - \alpha_1\, R_1 \tag{19}$$

After performing the row operation in Equation (19) above the augmented matrix row 3 becomes...

$$[R3C1] = 0 \text{ ...and... } [R3C2] = a_{32} - \alpha_1\, a_{12} \text{ ...and... } [R3C3] = a_{33} - \alpha_1\, a_{13} \text{ ...and... } [R3C4] = p_3 - \alpha_1\, p_1 \tag{20}$$

The augmented matrix after row operations Equation (17) and Equation (18) becomes...

$$\mathbf{A}' = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & p_1 \\ 0 & a_{22} - \alpha_1\, a_{12} & a_{23} - \alpha_1\, a_{13} & p_2 - \alpha_1\, p_1 \\ 0 & a_{32} - \alpha_1\, a_{12} & a_{33} - \alpha_1\, a_{13} & p_3 - \alpha_1\, p_1 \end{array}\right] \tag{21}$$

To make augmented matrix (as defined by Equation (21) above) element [R3C2] equal to zero we perform the following row operation...

$$R_3 = R_3 - \frac{a_{32} - \alpha_1\, a_{12}}{a_{22} - \alpha_1\, a_{12}} R_2 = R_3 - \alpha_2\, R_2 \tag{22}$$

After performing the row operation in Equation (22) above the augmented matrix row 3 becomes...

$$\begin{aligned} &[R3C1] = 0 \text{ ...and... } [R3C2] = 0 \text{ ...and... } [R3C3] = a_{33} - \alpha_1\, a_{13} - \alpha_2(a_{23} - \alpha_1\, a_{13}) \\ &\text{...and... } [R3C4] = p_3 - \alpha_1\, p_1 - \alpha_2(p_2 - \alpha_1\, p_1) \end{aligned} \tag{23}$$

The augmented matrix after row operation Equation (23) becomes...

$$\mathbf{A}' = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & p_1 \\ 0 & a_{22} - \alpha_1\, a_{12} & a_{23} - \alpha_1\, a_{13} & p_2 - \alpha_1\, p_1 \\ 0 & 0 & a_{33} - \alpha_1\, a_{13} - \alpha_2(a_{23} - \alpha_1\, a_{13}) & p_3 - \alpha_1\, p_1 - \alpha_2(p_2 - \alpha_1\, p_1) \end{array}\right] \tag{24}$$

To solve the system of linear equations as defined by Equation (24) above, which is matrix A as defined by Equation (1) in row echelon form, we first solve for θ_y (i.e. matrix row 3), which is the number of units of asset Y that we either purchase or sell short at time zero. This statement in equation form is...

$$(a_{33} - \alpha_1\, a_{13} - \alpha_2(a_{23} - \alpha_1\, a_{13}))\, \theta_y = p_3 - \alpha_1\, p_1 - \alpha_2(p_2 - \alpha_1\, p_1) \tag{25}$$

Using Appendix Equations (38), (39), (40), (41) and (42) the solution to the left side of Equation (25) is...

$$\begin{aligned}(a_{33} - \alpha_1\, a_{13} - \alpha_2(a_{23} - \alpha_1\, a_{13}))\, \theta_y &= (X_a Y_b + \alpha_1 X_a Y_0 + \alpha_1 X_0 Y_b - X_b Y_a - \alpha_1 X_b Y_0 - \alpha_1 X_0 Y_a)\, \theta_y \\ &= (X_a Y_b + (-X_a Y_a\, q - X_a Y_b + X_a Y_b\, q) + (-X_a Y_b\, q - X_b Y_b + X_b Y_b\, q) - X_b Y_a \\ &\quad - (-X_b Y_a\, q - X_b Y_b + X_b Y_b\, q) - (-X_a Y_a\, q - X_b Y_a + X_b Y_a\, q))\, \theta_y \\ &= (0)\,(\theta_y) \\ &= 0 \end{aligned} \tag{26}$$

Since $p_1 = 0$ the solution to the right side of Equation (25) is...

$$p_3 - \alpha_1\, p_1 - \alpha_2(p_2 - \alpha_1\, p_1) = p_3 - \alpha_2 p_2 \tag{27}$$

Using Appendix Equation (36) we can redefine α_2 (as originally defined by Equation (16)) as...

$$\begin{aligned}\alpha_2 &= \Big(X_b + \alpha_1 X_0\Big) \div \Big(X_a + \alpha_1 X_0\Big) \\ &= \Big(X_b - \frac{B_T}{B_0} X_0\Big) \div \Big(X_a - \frac{B_T}{B_0} X_0\Big) \\ &= \Big(X_b - \mathbb{E}^Q[X_T]\Big) \div \Big(X_a - \mathbb{E}^Q[X_T]\Big)\end{aligned} \tag{28}$$

Note that α_2 will always be a negative number since the expected value of X_a and X_b will always be between X_a and X_b, which means that (1) the numerator is positive and the denominator is negative or (2) the numerator is negative and the denominator is positive. If $\alpha_2 < 0$, $p_1 = 0$, $p_2 \geq 0$, $p_3 \geq 0$ and $p_2 + p_3 > 0$ then we can make the following statement as to the value of the right side of Equation (25)...

$$p_3 - \alpha_1\, p_1 - \alpha_2(p_2 - \alpha_1\, p_1) \neq 0 \tag{29}$$

Conclusion: Since the left side of Equation (25) is zero (see Equation (26)) and the right side is non-zero (see Equations (27), (28) and (29)) then Equation (25) cannot be solved and therefore not only is there no solution to the arbitrage as defined by vector u, there are no arbitrage possibilities at all. The last part of our plan has been accomplished!

Which leads us to...

The Fundamental Theorem of Finance - If a market is arbitrage-free then there exists a risk-neutral probability measure under which all assets earn the risk-free rate.

The Answer To Our Hypothetical Problem

Per Equation (11) above the equation for the price of Asset X using the risk-neutral probability measure Q is...

$$X_0 = \mathbb{E}^Q\Big[X_T \times \Big(1 + K_b\Big)^{-1}\Big] = \Big(X_a\, q + X_b\,(1 - q)\Big)\Big(1 + K_b\Big)^{-1} \tag{30}$$

According to our hypothetical problem we are given the prices of assets B and X at time zero. To determine the risk-neutral probability Measure Q we will use Equation (30) above and solve for q. Because asset B is risk-free using that asset to determine the risk-neutral probability Measure Q will not work. Using Equation (30) and information from Tables 1 and 3 the equation for the risk-neutral probability q is...

$$q = \frac{X_0(1+K_b)}{X_a - X_b} = \frac{(80)(1+0.05)}{120-60} = 0.40 \tag{31}$$

The table below presents state probabilities under the actual probability Measure P (from Table 2) and the risk-neutral probability Measure Q (from Equation (31))...

Table 5 - Probability Measures P (Actual) and Q (Risk-Neutral)

Description	Measure P	Measure Q
Probability that we will find ourselves in state ω_a at time one	0.60	0.40
Probability that we will find ourselves in state ω_b at time one	0.40	0.60

Using risk-neutral pricing Equation (12) and the risk-neutral probability measure Q as defined by Equation (31) the price of Asset Y that prohibits arbitrage (and is the answer to our hypothetical problem) is...

$$Y_0 = \Big(Y_a\, q + Y_b\,(1-q)\Big)\Big(1+K_b\Big)^{-1} = \Big((40)(0.40) + (80)(1-0.40)\Big)\Big(1+0.05\Big)^{-1} = 60.95 \tag{32}$$

Summary: If the price of Asset Y at time zero is \$60.95 then are no arbitrage opportunities in our two state economy.

Appendix

A. Using risk-neutral pricing Equation (10) we can write the following equation as...

$$\begin{aligned}
B_0\Big(X_aY_b - X_bY_a\Big) &= \Big(B_a\, q + B_b\,(1-q)\Big)\Big(X_aY_b - X_bY_a\Big)\Big(1+K_b\Big)^{-1} \\
&= \Big(B_a\, q + B_b - B_b\, q)\Big)\Big(X_aY_b - X_bY_a\Big)\Big(1+K_b\Big)^{-1} \\
&= \Big(B_aX_aY_b\, q - B_aX_bY_a\, q + B_bX_aY_b - B_bX_bY_a - B_bX_aY_b\, q + B_bX_bY_a\, q\Big)\Big(1+K_b\Big)^{-1}
\end{aligned} \tag{33}$$

B. Using risk-neutral pricing Equation (11) we can write the following equation as...

$$\begin{aligned}
X_0\Big(B_aY_b - B_bY_a\Big) &= \Big(X_a\, q + X_b\,(1-q)\Big)\Big(B_aY_b - B_bY_a\Big)\Big(1+K_b\Big)^{-1} \\
&= \Big(X_a\, q + X_b - X_b\, q\Big)\Big(B_aY_b - B_bY_a\Big)\Big(1+K_b\Big)^{-1} \\
&= \Big(B_aX_aY_b\, q - B_bX_aY_a\, q + B_aX_bY_b - B_bX_bY_a - B_aX_bY_b\, q + B_bX_bY_a\, q\Big)\Big(1+K_b\Big)^{-1}
\end{aligned} \tag{34}$$

C. Using risk-neutral pricing Equation (12) we can write the following equation as...

$$\begin{aligned}
Y_0\Big(B_aX_b - B_bX_a\Big) &= \Big(Y_a\, q + Y_b\,(1-q)\Big)\Big(B_aX_b - B_bX_a\Big)\Big(1+K_b\Big)^{-1} \\
&= \Big(Y_a\, q + Y_b - Y_b\, q\Big)\Big(B_aX_b - B_bX_a\Big)\Big(1+K_b\Big)^{-1} \\
&= \Big(B_aX_bY_a\, q - B_bX_aY_a\, q + B_aX_bY_b - B_bX_aY_b - B_aX_bY_b\, q + B_bX_aY_b\, q\Big)\Big(1+K_b\Big)^{-1}
\end{aligned} \tag{35}$$

D. We can redefine Equation (11), which is the risk-neutral pricing equation for asset X, as...

$$X_0 = \Big(X_a\, q + X_b\,(1-q)\Big)\Big(1+K_b\Big)^{-1} = \Big(X_a\, q + X_b - X_b\, q\Big)\frac{B_0}{B_T} \tag{36}$$

E. We can redefine Equation (12), which is the risk-neutral pricing equation for asset Y, as...

$$Y_0 = \Big(Y_a\, q + Y_b\,(1-q)\Big)\Big(1+K_b\Big)^{-1} = \Big(Y_a\, q + Y_b - Y_b\, q\Big)\frac{B_0}{B_T} \tag{37}$$

F. Using Equations (1) and (16) we can rewrite Equation (**??**) as...

$$\begin{aligned} a_{33} - \alpha_1\, a_{13} - \alpha_2(a_{23} - \alpha_1\, a_{13}) &= Y_b + \alpha_1 Y_0 - \left(\frac{X_b + \alpha_1 X_0}{X_a + \alpha_1 X_0}\right)\Big(Y_a + \alpha_1 Y_0\Big) \\ &= (X_a + \alpha_1 X_0)(Y_b + \alpha_1 Y_0) - (X_b + \alpha_1 X_0)(Y_a + \alpha_1 Y_0) \\ &= X_a Y_b + \alpha_1 X_a Y_0 + \alpha_1 X_0 Y_b + \alpha_1^2 X_0 Y_0 - X_b Y_a - \alpha_1 X_b Y_0 - \alpha_1 X_0 Y_a - \alpha_1^2 X_0 Y_0 \\ &= X_a Y_b + \alpha_1 X_a Y_0 + \alpha_1 X_0 Y_b - X_b Y_a - \alpha_1 X_b Y_0 - \alpha_1 X_0 Y_a \end{aligned} \tag{38}$$

G. Using Equation (38) we can rewrite the following equation as...

$$\alpha_1 X_a Y_0 = -\frac{B_T}{B_0}\, X_a \Big(Y_a\, q + Y_b - Y_b\, q\Big)\frac{B_0}{B_T} = -X_a Y_a\, q - X_a Y_b + X_a Y_b\, q \tag{39}$$

H. Using Equation (38) we can rewrite the following equation as...

$$\alpha_1 X_0 Y_b = -\frac{B_T}{B_0}\, Y_b \Big(X_a\, q + X_b - X_b\, q\Big)\frac{B_0}{B_T} = -X_a Y_b\, q - X_b Y_b + X_b Y_b\, q \tag{40}$$

I. Using Equation (38) we can rewrite the following equation as...

$$\alpha_1 X_b Y_0 = -\frac{B_T}{B_0}\, X_b \Big(Y_a\, q + Y_b - Y_b\, q\Big)\frac{B_0}{B_T} = -X_b Y_a\, q - X_b Y_b + X_b Y_b\, q \tag{41}$$

J. Using Equation (38) we can rewrite the following equation as...

$$\alpha_1 X_0 Y_a = -\frac{B_T}{B_0}\, Y_a \Big(X_a\, q + X_b - X_b\, q\Big)\frac{B_0}{B_T} = -X_a Y_a\, q - X_b Y_a + X_b Y_a\, q \tag{42}$$